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On the Almost $\eta-$Ricci Solitons on Pseudosymmetric Lorentz Generalized Sasakian Space Forms

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Abstract

In this paper, we consider Lorentz generalized Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz generalized Sasakian space forms admitting $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz generalized Sasakian space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

1. Introduction

The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poincare conjecture in [1,2]. The Ricci flow is a flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(g(t)).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [3], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Ashoka et al. in [4,5], Bagewadi et al. in [6], Ingalahalli in [7], Bejan and Crasmareanu in [8], Blaga in [9], Chandra et al. in [10], Chen and Deshmukh in [11], Deshmukh et al. in [12], He and Zhu [13], Atçeken et al. in [14], Nagaraja and Premalatta in [15], Tripathi in [16] and many others.

$\phi-$sectional curvature plays the important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [17]. P. Alegre and D. Blair described generalized Sasakian space forms [18]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [19]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity.

In this paper, we consider Lorentz generalized Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz generalized Sasakian space forms admitting $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz generalized Sasakian space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

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2. Preliminaries

Let $\tilde{M}$ be a $(2n+1)-$dimensional semi-Riemannian manifold. If the $\tilde{M}$ semi-Riemannian manifold with (ϕ,ξ,η,g) structure tensors satisfies the following conditions, this manifold is called $\varepsilon-$ almost contact metric manifold and (ϕ,ξ,η) triple is called almost contact structure.

$$\begin{aligned}&\phi\xi=0,\quad \eta\circ\phi=0,\quad \eta(\xi)=1,\quad \phi^2=-Id+\eta\otimes\xi,\\ &g(Y_1,Y_2)=g(\phi Y_1,\phi Y_2)+\varepsilon\eta(Y_1)\eta(Y_2),\quad \eta(Y_1)=\varepsilon g(Y_1,\xi)\end{aligned}$$

where

$$\varepsilon=g(\xi,\xi)=\pm1.$$

If $d\eta$ and g provide the relation

$$d\eta(Y_1,Y_2)=g(Y_1,\phi Y_2)$$

then $\tilde{M}$ is called a contact pseudometric manifold and the (ϕ,ξ,η) triple is called a contact structure.
Let be define a $\left(h\left(\frac{d}{dY_1}\right),Y_2\right)$ vector field on $\mathbb{R}\times\tilde{M}$, where Y_1 is a coordinate on $\mathbb{R}$ and h is a C^∞ function on $\mathbb{R}\times\tilde{M}$. The structure defined as

$$J\left(h\frac{d}{dY_1},Y_2\right)=\left(\eta(Y_2)\frac{d}{dY_1},\phi Y_2-h\xi\right)$$

on $\mathbb{R}\times\tilde{M}$ is called a almost complex structure and $J^2=-id$. If J is integrable, the almost contact structure (ϕ,ξ,η) is said to be normal. If Y_1 is perpendicular to ξ, the plane spanned by Y_1 and ϕY_1, is called the $\phi-$section. The curvature of the ϕ-section is called the ϕ-sectional curvature. The curvature of the indefinite Sasakian manifold defined in this way is precisely determined by the $\phi-$section curvature. If the $\phi-$section curvature of the indefinite Sasakian manifold is equal to a constant c, the curvature tensor of this manifold is defined as

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3=&\left(\frac{c+3\varepsilon}{4}\right)\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}+\left(\frac{c-\varepsilon}{4}\right)\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1+2g(Y_1,\phi Y_2)\phi Y_3\}\\ &+\left(\frac{c-\varepsilon}{4}\right)\{\eta(Y_1)\eta(Y_3)Y_2-\eta(Y_2)\eta(Y_3)Y_1+\varepsilon g(Y_1,Y_3)\eta(Y_2)\xi-\varepsilon g(Y_2,Y_3)\eta(Y_1)\xi\}.\end{aligned}$$

For an $\varepsilon-$almost contact metric manifold $\tilde{M}$, if there are $F_1,F_2,F_3\in C^\infty\left(\tilde{M}\right)$ functions such that

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3=&F_1\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}+F_2\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1+2g(Y_1,\phi Y_2)\phi Y_3\}\\ &+F_3\{\eta(Y_1)\eta(Y_3)Y_2-\eta(Y_2)\eta(Y_3)Y_1+\varepsilon g(Y_1,Y_3)\eta(Y_2)\xi-\varepsilon g(Y_2,Y_3)\eta(Y_1)\xi\}\end{aligned}$$

then manifold $\tilde{M}$ is called a generalized indefinite Sasakian space form.
In this article, only the Lorentzian case, which corresponds to the $\varepsilon=-1$, where the index of the metric is 1, will be discussed. Such manifolds are called Lorentz generalized Sasakian space forms and are denoted by $M^{2n+1}(F_1,F_2,F_3)$. Thus, the curvature tensor of a $(2n+1)-$dimensional Lorentz generalized Sasakian space form is defined as

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3=&F_1\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}+F_2\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1+2g(Y_1,\phi Y_2)\phi Y_3\}\\ &+F_3\{\eta(Y_1)\eta(Y_3)Y_2-\eta(Y_2)\eta(Y_3)Y_1-g(Y_1,Y_3)\eta(Y_2)\xi+g(Y_2,Y_3)\eta(Y_1)\xi\}.\end{aligned}\tag{2.1}$$

Lemma 2.1. *Let $M^{2n+1}(F_1,F_2,F_3)$ be the $(2n+1)-$dimensional Lorentz generalized Sasakian space form. The following relations are provided for $M^{2n+1}(F_1,F_2,F_3)$.*

$$\tilde{\nabla}_{Y_1}\xi=(F_1+F_3)\phi Y_1,\tag{2.2}$$
$$\tilde{R}(Y_1,\xi)Y_3=-(F_1+F_3)[g(Y_1,Y_3)\xi+\eta(Y_3)Y_1],\tag{2.3}$$
$$\tilde{R}(\xi,Y_2)Y_3=(F_1+F_3)[g(Y_2,Y_3)\xi+\eta(Y_3)Y_2],\tag{2.4}$$
$$\tilde{R}(Y_1,Y_2)\xi=(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1],\tag{2.5}$$
$$\eta\left(\tilde{R}(Y_1,Y_2)Y_3\right)=(F_1+F_3)g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3),\tag{2.6}$$
$$S(Y_1,Y_2)=(2nF_1+3F_2+F_3)g(Y_1,Y_2)+(3F_2-(2n-1)F_3)\eta(Y_1)\eta(Y_2),\tag{2.7}$$
$$S(Y_1,\xi)=-2n(F_1+F_3)\eta(Y_1),\tag{2.8}$$
$$QY_1=(2nF_1+3F_2+F_3)Y_1+((2n-1)F_3-3F_2),\tag{2.9}$$
$$Q\xi=2n(F_1+F_3)\xi,\tag{2.10}$$

where $\tilde{R},S$ and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator of $M^{2n+1}(F_1,F_2,F_3)$, respectively.

Let M be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,...,X_K;Y_1,Y_2)=-T((Y_1\wedge_A Y_2)X_1,...,X_k)-...-T(X_1,...,X_{k-1},(Y_1\wedge_A Y_2)X_k),\tag{2.11}$$

where

$$(Y_1\wedge_A Y_2)Y_3=A(Y_2,Y_3)Y_1-A(Y_1,Y_3)Y_2,\tag{2.12}$$

$k \geq 1$, $X_1, X_2, ..., X_k, Y_1, Y_2 \in \Gamma(TM)$.
Precisely, a Ricci soliton on a Riemannian manifold $(\tilde{M}, g)$ is defined as a triple (g, ξ, λ) on $\tilde{M}$ satisfying

$$L_\xi g + 2S + 2\lambda g = 0 \tag{2.13}$$

where L_ξ is the Lie derivative operator along the vector field ξ and λ is a real constant. We note that if ξ is a Killing vector field, then the Ricci soliton reduces to an Einstein metric (g, λ). Futhermore, in [20], generalization is the notion of $\eta-$Ricci soliton defined by J.T. Cho and M. Kimura as a quadruple (g, ξ, λ, μ) satisfying

$$L_\xi g + 2S + 2\lambda g + 2\mu \eta \oplus \eta = 0 \tag{2.14}$$

where λ and μ are real constants and η is the dual of ξ and S denotes the Ricci tensor of $\tilde{M}$. Furthermore if λ and μ are smooth functions on $\tilde{M}$, then it called almost $\eta-$Ricci soliton on $\tilde{M}$ [20].
Suppose the quartet (g, ξ, λ, μ) is almost $\eta-$Ricci soliton on manifold $\tilde{M}$. Then,

- If $\lambda < 0$, then $\tilde{M}$ is shriking.
- If $\lambda = 0$, then $\tilde{M}$ is steady.
- If $\lambda > 0$, then $\tilde{M}$ is expanding.

3. Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric and Ricci Semisymmetric Lorentz Generalized Sasakian Space Forms

Now let (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form. Then we have

$$\begin{aligned}\left(L_\xi g\right)(Y_1, Y_2) =& L_\xi g(Y_1, Y_2) - g\left(L_\xi Y_1, Y_2\right) - g\left(Y_1, L_\xi Y_2\right) \\ =& \xi g(Y_1, Y_2) - g([\xi, Y_1], Y_2) - g(Y_1, [\xi, Y_2]) \\ =& g\left(\nabla_\xi Y_1, Y_2\right) + g\left(Y_1, \nabla_\xi Y_2\right) - g\left(\nabla_\xi Y_1, Y_2\right) + g(\nabla_{Y_1}\xi, Y_2) - g\left(\nabla_\xi Y_2, Y_1\right) + g(Y_1, \nabla_{Y_2}\xi),\end{aligned}$$

for all $Y_1, Y_2 \in \Gamma(TM)$. By using ϕ is anti-symmetric, we have

$$\left(L_\xi g\right)(Y_1, Y_2) = 0. \tag{3.1}$$

Thus, in a Lorentz generalized Sasakian space form, from (2.14) and (3.1), we have

$$S(Y_1, Y_2) + \lambda g(Y_1, Y_2) + \mu \eta(Y_1) \eta(Y_2) = 0. \tag{3.2}$$

It is clear from (16) that the $(2n+1)-$dimensional Lorentz generalized Sasakian admitting almost $\eta-$Ricci soliton $\left(M^{2n+1}, g, \xi, \lambda, \mu\right)$ is an $\eta-$Einstein manifold.
For $Y_2 = \xi$ in (3.2), this implies that

$$S(\xi, Y_1) = (\lambda - \mu) \eta(Y_1). \tag{3.3}$$

Taking into account of (3.3), we conclude that

$$\mu - \lambda = 2n(F_1 + F_3).$$

Definition 3.1. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $\tilde{R} \cdot S$ and $Q(g, S)$ are linearly dependent, then the M^{2n+1} is said to be* ***Ricci pseudosymmetric****.*

In this case, there exists a function L_1 on M^{2n+1} such that

$$\tilde{R} \cdot S = L_1 Q(g, S).$$

In particular, if $L_1 = 0$, the manifold M^{2n+1} is said to be **Ricci semisymmetric**.
Let us now investigate the Ricci pseudosymmetric case of the $(2n+1)-$dimensional Lorentz generalized Sasakian space forms.

Theorem 3.2. *Let M^{2n+1} be Lorentz generalized Sasakian space forms and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci pseudosymmetric, then*

$$L_1 = \frac{(F_1 + F_3)[\lambda - 2n(F_1 + F_3)]}{\mu}$$

provided $\mu \neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space forms M^{2n+1}. That is mean

$$\left(\tilde{R}(Y_1, Y_2) \cdot S\right)(Y_4, Y_5) = L_1 Q(g, S)(Y_4, Y_5; Y_1, Y_2),$$

for all $Y_1, Y_2, Y_4, Y_5 \in \Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S\left(\tilde{R}(Y_1, Y_2)Y_4, Y_5\right) + S\left(Y_4, \tilde{R}(Y_1, Y_2)Y_5\right) = L_1\left\{S\left((Y_1 \wedge_g Y_2)Y_4, Y_5\right) + S\left(Y_4, (Y_1 \wedge_g Y_2)Y_5\right)\right\}. \tag{3.4}$$

If we choose $Y_5=\xi$ in (3.4), we get

$$S\left(\tilde{R}(Y_1,Y_2)Y_4,\xi\right)+S\left(Y_4,\tilde{R}(Y_1,Y_2)\xi\right)=L_1\left\{S\left(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi\right)\ +S\left(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1\right)\right\}. \tag{3.5}$$

If we make use of (2.5) and (2.8) in (3.5), we have

$$\begin{aligned} &S\left(Y_4,(F_1+F_3)\left[\eta(Y_2)Y_1-\eta(Y_1)Y_2\right]\right)-2n(F_1+F_3)\eta\left(\tilde{R}(Y_1,Y_2)Y_4\right)\\ &=L_1\left\{-2n(F_1+F_3)g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\ +S\left(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1\right)\right\}. \end{aligned} \tag{3.6}$$

If we use (2.6) in the (3.6), we get

$$\begin{aligned} &-2n(F_1+F_3)^2 g\left(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4\right)+(F_1+F_3)S\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\\ &=L_1\left\{-2n(F_1+F_3)g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\ +S\left(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1\right)\right\}. \end{aligned}$$

If we use (3.2) in (3.5), we can write

$$\left[(F_1+F_3)\left[2n(F_1+F_3)-\lambda\right]+\left[\lambda+2n(F_1+F_3)\right]L_1\right]\times g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)=0. \tag{3.7}$$

It is clear from (3.7)

$$L_1=\frac{(F_1+F_3)\left[\lambda-2n(F_1+F_3)\right]}{\lambda+2n(F_1+F_3)}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.3. *Let M^{2n+1} be a Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci semisymmetric, then $\lambda=2n(F_1+F_3)$ and $\mu=4n(F_1+F_3)$.*

Corollary 3.4. *Let M^{2n+1} be a Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci semisymmetric, then M^{2n+1} is an $\eta-$Einstein manifold.*

Corollary 3.5. *Let M^{2n+1} be a Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci semisymmetric, then we observe that:*

(i) M^{2n+1} is expanding, if $F_1+F_3>0$.
(ii) M^{2n+1} is shriking, if $F_1+F_3<0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the concircular curvature tensor is defined as

$$C(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{r}{2n(2n+1)}\left[g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\right]. \tag{3.8}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.8), we can write

$$C(Y_1,Y_2)\xi=\left[(F_1+F_3)-\frac{r}{2n(2n+1)}\right]\left[\eta(Y_1)Y_2-\eta(Y_2)Y_1\right], \tag{3.9}$$

and similarly if we take the inner product of both sides of (24) by ξ, we get

$$\eta\left(C(Y_1,Y_2)Y_3\right)=\left[(F_1+F_3)-\frac{r}{2n(2n+1)}\right]g\left(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3\right). \tag{3.10}$$

Definition 3.6. *Let M^{2n+1} be a $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $C\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be **concircular Ricci pseudosymmetric.***

In this case, there exists a function L_2 on M^{2n+1} such that

$$C\cdot S=L_2Q(g,S).$$

In particular, if $L_2=0$, the manifold M^{2n+1} is said to be **concircular Ricci semisymmetric.**
Let us now investigate the concircular Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.7. *Let M^{2n+1} be a Lorentz generalized Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a concircular Ricci pseudosymmetric, then*

$$L_2=\frac{\left[\lambda-2n(F_1+F_3)\right]\left[2n(2n+1)(F_1+F_3)-r\right]}{2n(2n+1)\mu},$$

provided $\mu\neq0$.

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be concircular Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. That is mean

$$(C(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_2Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(C(Y_1,Y_2)Y_4,Y_5)+S(Y_4,C(Y_1,Y_2)Y_5)=L_2\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.11}$$

If we choose $Y_5=\xi$ in (3.11), we get

$$S(C(Y_1,Y_2)Y_4,\xi)+S(Y_4,C(Y_1,Y_2)\xi)=L_2\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.12}$$

By using of (2.8) and (3.9) in (3.12), we have

$$\begin{aligned}&S(Y_4,A[\eta(Y_1)Y_2-\eta(Y_2)Y_1])-2n(F_1+F_3)\eta(C(Y_1,Y_2)Y_4)\\&=L_2\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\},\end{aligned} \tag{3.13}$$

where $A=(F_1+F_3)-\frac{r}{2n(2n+1)}$. Substituting (3.10) into (3.13), we have

$$\begin{aligned}&-2n(F_1+F_3)Ag(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)+AS(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=L_2\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}.\end{aligned} \tag{3.14}$$

If we use (3.2) in the (3.14), we can write

$$\{A[2n(F_1+F_3)-\lambda]+[\lambda+2n(F_1+F_3)]L_2\}\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.15}$$

It is clear from (3.15),

$$L_2=\frac{[\lambda-2n(F_1+F_3)][2n(2n+1)(F_1+F_3)-r]}{2n(2n+1)\mu}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.8. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a concircular Ricci semisymmetric, then M^{2n+1} is either manifold with scalar curvature $r=2n(2n+1)(F_1+F_3)$ or $\lambda=2n(F_1+F_3)$.*

Corollary 3.9. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a concircular Ricci semisymmetric, then we observe that:*

(i) The soliton M^{2n+1} is expanding, if $(F_1+F_3)>0$.
(ii) The soliton M^{2n+1} is shriking, if $(F_1+F_3)<0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the projective curvature tensor is defined as

$$P(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{1}{2n}[S(Y_2,Y_3)Y_1-S(Y_1,Y_3)Y_2]. \tag{3.16}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.16) we can write

$$P(Y_1,Y_2)\xi=0, \tag{3.17}$$

and similarly if we take the inner product of both sides of (3.16) by ξ, we get

$$\eta(P(Y_1,Y_2)Y_3)=0. \tag{3.18}$$

Definition 3.10. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $P\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be* ***projective Ricci pseudosymmetric.***

In this case, there exists a function L_3 on M^{2n+1} such that

$$P\cdot S=L_3Q(g,S).$$

In particular, if $L_3=0$, the manifold M^{2n+1} is said to be **projective Ricci semisymmetric.**
Let us now investigate the projective Ricci pseudosymmetry case of the Lorentz generalized Sasakian space form.

Theorem 3.11. *Let M^{2n+1} be Lorentz Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a projective Ricci pseudosymmetric, then M^{2n+1} is either projective Ricci semisymmetric or almost $\eta-$Ricci soliton (g,ξ,λ,μ) reduces almost Ricci soliton (g,ξ,λ).*

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. Then we have

$$(P(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_3Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(P(Y_1,Y_2)Y_4,Y_5)+S(Y_4,P(Y_1,Y_2)Y_5)=L_3\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.19}$$

If we choose $Y_5=\xi$ in (3.19), we get

$$S(P(Y_1,Y_2)Y_4,\xi)+S(Y_4,P(Y_1,Y_2)\xi)=L_3\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.20}$$

If we make use of (2.8) and (3.17) in (3.20) we have

$$-2n(F_1+F_3)\eta(P(Y_1,Y_2)Y_4)=L_3\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.21}$$

If we use (3.18) in the (3.21), we get

$$L_3\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}=0. \tag{3.22}$$

If we use (3.2) in the (3.22), we can write

$$L_3[\lambda+2n(F_1+F_3)]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.23}$$

It is clear from (3.23),

$$\mu L_3=0.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.12. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a projective Ricci pseudosymmetric, then M^{2n+1} is either projective Ricci semisymmetric or we observe that:*

(i) The soliton M^{2n+1} is expanding, if $F_1+F_3<0$.
(ii) The soliton M^{2n+1} is shriking, if $F_1+F_3>0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the $\mathscr{M}-$projective curvature tensor is defined as

$$\mathscr{M}(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\tfrac{1}{2n}[S(Y_2,Y_3)Y_1-S(Y_1,Y_3)Y_2\ +g(Y_2,Y_3)QY_1-g(Y_1,Y_3)QY_2] \tag{3.24}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.24), we obtain

$$\mathscr{M}(Y_1,Y_2)\xi=\frac{1}{2n}[\eta(Y_2)QY_1-\eta(Y_1)QY_2], \tag{3.25}$$

and similarly if we take the inner product of both of sides of (3.24) by ξ, we get

$$\eta(\mathscr{M}(Y_1,Y_2)Y_3)=\frac{1}{2n}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3). \tag{3.26}$$

Definition 3.13. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $\mathscr{M}\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be $\mathscr{M}-$**projective Ricci pseudosymmetric.***

In this case, there exists a function L_4 on M^{2n+1} such that

$$\mathscr{M}\cdot S=L_4Q(g,S).$$

In particular, if $L_4=0$, the manifold M^{2n+1} is said to be $\mathscr{M}-$**projective Ricci semisymmetric.**
Let us now investigate the $\mathscr{M}-$projective Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.14. *Let M^{2n+1} be Lorentz generalized Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a $\mathscr{M}-$projective Ricci pseudosymmetric, then*

$$L_4=\frac{\lambda^2-2n(F_1+F_3)\lambda}{2n\mu},$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be projective $\mathscr{M}-$projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. That is mean

$$(\mathscr{M}(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_4Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(\mathscr{M}(Y_1,Y_2)Y_4,Y_5)+S(Y_4,\mathscr{M}(Y_1,Y_2)Y_5)=L_4\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.27}$$

If we choose $Y_5=\xi$ in (3.27) we get

$$S(\mathscr{M}(Y_1,Y_2)Y_4,\xi)+S(Y_4,\mathscr{M}(Y_1,Y_2)\xi)=L_4\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.28}$$

If we make use of (2.8) and (3.25) in (3.28), we have

$$\begin{aligned}&-2n(\digamma_1+\digamma_3)\eta(\mathscr{M}(Y_1,Y_2)Y_4)+S\left(Y_4,\frac{1}{2n}[\eta(Y_2)QY_1-\eta(Y_1)QY_2]\right)\\&=L_4\{-2n(\digamma_1+\digamma_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}.\end{aligned} \tag{3.29}$$

By using (3.26) in the (3.29), we get

$$\begin{aligned}&-(\digamma_1+\digamma_3)S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)+\frac{1}{2n}S(\eta(Y_2)QY_1-\eta(Y_1)QY_2,Y_4)\\&=L_4\{-2n(\digamma_1+\digamma_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}.\end{aligned} \tag{3.30}$$

If we put (3.2) in (3.30), we can write

$$\begin{aligned}&\lambda(\digamma_1+\digamma_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)-\frac{\lambda}{2n}S(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)\\&=L_4[\lambda+2n(\digamma_1+\digamma_3)]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned} \tag{3.31}$$

Again, if we use (3.2) in the (3.31), we obtain

$$\left\{\tfrac{\lambda^2}{2n}-(\digamma_1+\digamma_3)\lambda-L_4[\lambda+2n(\digamma_1+\digamma_3)]\right\}\times g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)=0. \tag{3.32}$$

It is clear from (3.32),

$$L_4=\frac{\lambda^2-2n(\digamma_1+\digamma_3)\lambda}{2n[2n(\digamma_1+\digamma_3)+\lambda]}.$$

This completes the proof. □

We can give the following corollaries.

Corollary 3.15. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a $\mathscr{M}-$projective Ricci semisymmetric, then M^{2n+1} is either steady or $\eta-$Einstein manifold.*

Corollary 3.16. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a $\mathscr{M}-$projective Ricci semisymmetric, then M^{2n+1} is either steady or we observe that:*

(i) The soliton M^{2n+1} is shriking if λ is between 0 and $2n(\digamma_1+\digamma_3)$.
(ii) The soliton M^{2n+1} is steady if $\lambda=0$.
(iii) The soliton M^{2n+1} is expanding for other cases of λ.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the W_1-curvature tensor is defined as

$$W_1(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3+\frac{1}{2n}[S(Y_2,Y_3)Y_1-S(Y_1,Y_3)Y_2]. \tag{3.33}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.33), we can write

$$W_1(Y_1,Y_2)\xi=2(\digamma_1+\digamma_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1], \tag{3.34}$$

and similarly if we take the inner product of both of sides of (3.33) by ξ, we get

$$\eta(W_1(Y_1,Y_2)Y_3)=2(\digamma_1+\digamma_3)g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3). \tag{3.35}$$

Definition 3.17. *Let M^{2n+1} be a $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $W_1\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1-**Ricci pseudosymmetric.***

In this case, there exists a function L_5 on M^{2n+1} such that

$$W_1\cdot S=L_5Q(g,S).$$

In particular, if $L_5=0$, the manifold M^{2n+1} is said to be W_1-**Ricci semisymmetric.**
Let us now investigate the W_1-Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.18. *Let M^{2n+1} be Lorentz generalized Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci pseudosymmetric, then*

$$L_5=\frac{2(F_1+F_3)[\lambda-2n(F_1+F_3)]}{\mu}$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz Sasakian space form M^{2n+1} be W_1-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. That is mean

$$(W_1(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_5Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(W_1(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_1(Y_1,Y_2)Y_5)=L_5\left\{S\left((Y_1\wedge_g Y_2)Y_4,Y_5\right)+S\left(Y_4,(Y_1\wedge_g Y_2)Y_5\right)\right\}. \tag{3.36}$$

If we choose $Y_5=\xi$ in (3.36), we get

$$S(W_1(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_1(Y_1,Y_2)\xi)=L_5\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.37}$$

If we make use of (2.8) and (3.34) in (3.37) we have

$$\begin{aligned}&2(F_1+F_3)S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)-2n(F_1+F_3)\eta(W_1(Y_1,Y_2)Y_4)\\&=L_5\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}.\end{aligned} \tag{3.38}$$

If we use (3.35) in the (3.38), we get

$$\begin{aligned}&-4n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)+2(F_1+F_3)S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=L_5\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)+S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}.\end{aligned} \tag{3.39}$$

If we use (3.2) in the (3.39), we can write

$$\{2(F_1+F_3)[2n(F_1+F_3-\lambda)]+L_5[\lambda+2n(F_1+F_3)]\}\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0 \tag{3.40}$$

It is clear from (3.40),

$$L_5=\frac{2(F_1+F_3)[\lambda-2n(F_1+F_3)]}{\lambda+2n(F_1+F_3)}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.19. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci semisymmetric, then $\lambda=2n(F_1+F_3)$ provided $\mu\neq 0$.*

Corollary 3.20. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci semisymmetric, then we observe that:*

(i) The soliton M^{2n+1} is expanding, if $(F_1+F_3)>0$.
(ii) The soliton M^{2n+1} is shriking, if $(F_1+F_3)<0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the W_2-curvature tensor is defined as

$$W_2(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{1}{2n}[g(Y_2,Y_3)QY_1-g(Y_1,Y_3)QY_2]. \tag{3.41}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.41), we can write

$$W_2(Y_1,Y_2)\xi=(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1]-\tfrac{1}{2n}[\eta(Y_1)QY_2-\eta(Y_2)QY_1], \tag{3.42}$$

and similarly if we take the inner product of both sides of (3.41) by ξ, we get

$$\eta(W_2(Y_1,Y_2)Y_3)=(F_1+F_3)g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3)+\tfrac{1}{2n}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3). \tag{3.43}$$

Definition 3.21. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $W_2\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_2-**Ricci pseudosymmetric.***

In this case, there exists a function L_6 on M^{2n+1} such that

$$W_2\cdot S=L_6Q(g,S).$$

In particular, if $L_6=0$, the manifold M^{2n+1} is said to be W_2-**Ricci semisymmetric.**
Let us now investigate the W_2-Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.22. *Let M^{2n+1} be Lorentz generalized Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_2-Ricci pseudosymmetric, then*

$$L_6=-\frac{\lambda^2+4n^2(F_1+F_3)^2}{2n\mu}$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form be W_2-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form. That is mean

$$(W_2(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_6Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(W_2(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_2(Y_1,Y_2)Y_5)=L_6\left\{S\left((Y_1\wedge_g Y_2)Y_4,Y_5\right)+S\left(Y_4,(Y_1\wedge_g Y_2)Y_5\right)\right\}. \tag{3.44}$$

If we choose $Y_5=\xi$ in (3.44), we get

$$S(W_2(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_2(Y_1,Y_2)\xi)=L_6\left\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_2)Y_1-\eta(Y_1)Y_2)\right\}. \tag{3.45}$$

If we make use of (2.8) and (3.42) in (3.45), we have

$$\begin{aligned}&-2n(F_1+F_3)\eta(W_2(Y_1,Y_2)Y_4)+S\left(Y_4,(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1]-\frac{1}{2n}[\eta(Y_1)QY_2-\eta(Y_2)QY_1]\right)\\&=L_6\left\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}.\end{aligned} \tag{3.46}$$

If we use (3.43) in the (3.46), we get

$$\begin{aligned}&-2n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)-\frac{1}{2n}S(Y_4,\eta(Y_1)QY_2-\eta(Y_2)QY_1)\\&=L_6\left\{S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\ -2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right\}.\end{aligned} \tag{3.47}$$

If we use (3.2) in the (3.47), we have

$$\begin{aligned}&-2n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)-\frac{\lambda}{2n}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=-L_6[\lambda+2n(F_1+F_3)]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned} \tag{3.48}$$

Again, if we use (3.2) in (3.48), we obtain

$$\left\{\tfrac{\lambda^2}{2n}+2n(F_1+F_3)^2+L_6[\lambda+2n(F_1+F_3)]\right\}\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.49}$$

It is clear from (3.49),

$$L_6=-\frac{\lambda^2+4n^2(F_1+F_3)^2}{2n[\lambda+2n(F_1+F_3)]}.$$

This completes the proof. □

4. Conclusion

In this paper, we consider Lorentz generalized Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz generalized Sasakian space forms admitting $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz generalized Sasakian space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

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